Newton's law of resistance which, as is well-known /4/, works much better for convex bodies than for concave ones. One can expect particularly large deviations from this theory in bodies with "inner positive" corners, such as the corner at the point d in configurations with rear vertical segments. The first way to guarantee a better approximation to reality is to introduce an additional restriction on the radius of curvature of the admissible contours: $R \geqslant r$, where $r > 0$ is some given constant. When that is done the corner at d is rounded off to radius r. Another method would be to introduce point forces at "positive" corners. The question of modifying the solution in the context of this approach requires additional anaysis.

The author is indebted to O.A. Gil'man, who drew attention to the problem, and to N.I. Tillyayeva and V.A. Vostretsov for their assistance.

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SELFSIMILAR SOLUTIONS DESCRIBING THERMAL CAPILLARY FLOWS IN VISCOUS LAYERS*

V.A. BATYSHCHEV

Thermal capillary flows in thin layers, brought about by non-uniform heating of the free boundary, are investigated at high Marangoni numbers. Selfsimilar solutions of the non-linear boundary-layer equations are constructed under conditions of axial symmetry, and asymptotic formulae for the solutions are found for small and large values of the thickness of the layer. It is shown that the selfsimilar solutions may not be unique when the parameters of the problem have certain values. The buoyancy forces in an inhomogeneous fluid lead to the reinforcement or suppression of the flows or to the formation of reverse flows close to the free boundary. Selfsimilar solutions when there are thermal capillary effects present have been studied in /1-5/.

1. The non-linear axially-symmetric problem of the stationary thermal capillary motion of an incompressible fluid in a thin layer, bounded by a free surface Γ and a solid wall S is considered at low coefficients of viscosity $\nu \to 0$ and thermal diffusivity $\chi \to 0$ when there is a zero temperature gradient on the free boundary:

$$(\mathbf{v}, \nabla)\,\mathbf{v} = -\rho^{-1}\nabla p + \nu\Delta\mathbf{v} + \mathbf{g} \quad (1.1)$$
$$\mathbf{v}\nabla T = \chi\Delta T, \ \operatorname{div}\mathbf{v} = 0$$

$$p = 2\nu\rho\mathbf{n}\Pi\mathbf{n} + \sigma(k_1 + k_2) + p_*, \ (x, y, z) \in \Gamma \quad (1.2)$$
$$2\nu\rho\,[\Pi\mathbf{n} - (\mathbf{n}\Pi\mathbf{n})] = \nabla_\Gamma\sigma, \ \mathbf{v}\mathbf{n} = 0, \ (x, y, z) \in \Gamma$$
$$T = T_\Gamma, \ (x, y, z) \in \Gamma; \ \mathbf{v} = T - T_S = 0, \ (x, y, z) \in S$$

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Here $\mathbf{v} = (v_x, v_y, v_z)$ is the velocity vector, T is the temperature, $\mathbf{g} = -g\mathbf{e}_z$, $\mathbf{e}_z = (0, 0, 1)$ is the unit vector along the z-axis, g is the acceleration due to gravity, $\mathbf{n}$ is the unit vector of the external normal to the free boundary Γ, Π is the rate of deformation tensor, k_1 and k_2 are the principal curvatures of the surface Γ, p_* and T_Γ are the specified temperature and pressure on Γ, T_S is the temperature of the wall, $\nabla_\Gamma = \nabla - (\mathbf{n}, \nabla)\,\mathbf{n}$ is the gradient along Γ and σ is the surface tension coefficient which is assumed to be a linear function of temperature: $\sigma = \sigma_0 - |\sigma_T|(T - T_*)$, where σ_0, σ_T and T_* are known constants and $\sigma_T < 0$.

When the free boundary is non-uniformly heated, tangential surface stresses arise along Γ which, when $\nu, \chi \to 0$, lead to the formation of non-linear boundary layers close to the boundaries of the domain as a consequence of the thermocapillary effect.

We shall reduce problem (1.1), (1.2) to dimensionless form by introducing the characteristic scales of length L, velocity $U = (|\sigma_T|^2 A^2 L \rho^{-2} \nu^{-1})^{1/3}$ and pressure $P = \sigma_0/L$, where A is the characteristic scale of the temperature gradient. We also introduce a small parameter $\varepsilon = M^{-1/3}$, where $M = |\sigma_T| L^2 A \rho^{-1} \nu^{-2}$ is the Marangoni number. We note that small values of ν or large temperature gradients correspond to small values of ε. It was shown in /5/ that the orders of the velocity and the thickness of the boundary layer close to the free boundary are equal to $O(U)$ and $O(\varepsilon)$ respectively.

Next, we shall consider thermocapillary flows in thin layers with a thickness of the order of ε. We note that flows in layers with a thickness ε which are bounded by two solid walls have been considered in /6/ and the pressure gradient was determined when solving the Prandtl equations.

Asymptotic expansions of the solution of problem (1.1), (1.2) are constructed in the form

$$\mathbf{v} \sim \mathbf{h}_0 + \varepsilon \mathbf{h}_1 + \ldots, \quad p' \sim \lambda q_0 + \varepsilon \lambda q_1 + \ldots \tag{1.3}$$
$$T \sim \theta_0 + \varepsilon \theta_1 + \ldots, \quad \zeta \sim \varepsilon \zeta_1 + \varepsilon^2 \zeta_2 + \ldots, \quad (\varepsilon \to 0), \quad p' = (p + \rho g z)/P$$

Here, $\lambda = |\sigma_T| AL/\sigma_0$ is the capillary constant and $z = \zeta(x, y)$ is the equation of the free boundary.

2. A boundary-value problem for determining the principal terms of the asymptotic forms (1.3) is obtained by applying a second iterative process using the Vishik-Lyusternik method /7/ to system (1.1), (1.2). The local orthogonal coordinates ξ, φ and θ, where ξ is the distance of the point $N(x, y, z)$ to the surface S and φ and θ are the curvilinear coordinates on S of the foot of the normal dropped from a point N. Let $h_{\varphi k}$, $h_{\theta k}$ and $h_{\xi k}$ be the components of a vector $\mathbf{h}_k$ in the local coordinates. We substitute (1.3) into (1.1), (1.2) and introduce the elongating transformation $\xi = \varepsilon s$. By equating the coefficients of ε^{-1} and ε to zero, we find that $h_{\xi 0} = 0$ and $h_{\varphi 0}$, $h_{\theta 0}$ and $h_{\xi 1}$ satisfy the Prandtl boundary-layer equations.

We will now present these equations in the axially-symmetric case by assuming that $h_{\theta 0} = 0$, the vector $\mathbf{v}$ is independent of the coordinate θ and φ is the length of an arc of the surface of rotation S in the meridional cross-section:

$$h_{\varphi 0} \frac{\partial h_{\varphi 0}}{\partial \varphi} + h_{\xi 1} \frac{\partial h_{\varphi 0}}{\partial s} = \frac{\partial^2 h_{\varphi 0}}{\partial s^2} - \frac{\partial q_0}{\partial s} \tag{2.1}$$
$$\frac{\partial q_0}{\partial s} = 0, \quad \frac{\partial}{\partial \varphi}(r h_{\varphi 0}) + \frac{\partial}{\partial s}(r h_{\xi 1}) = 0$$
$$\frac{\partial h_{\varphi 0}}{\partial s} = -\frac{\partial \sigma}{\partial \varphi}, \quad \mathbf{h}_1 \mathbf{n} = 0 \; (s = \zeta_1(\varphi)), \quad h_{\varphi 0} = h_{\xi 1} = 0 \quad (s = 0)$$

Here, $r(\varphi)$ is the distance from a point on the surface S to the axis of rotation z.

We supplement system (2.1) with an equation for determining the function $\zeta_1(\varphi)$, using dynamic boundary conditions on the free boundary. We shall consider the case when the capillary constant is of the order of ε^2, that is, $\lambda = \lambda_0 \varepsilon^2$. When account is taken of the properties of axial symmetry, the boundary conditions (1.2) for the normal stresses now lead to the equations

$$\lambda_0 q_0 = B \zeta_1 \cos(\mathbf{n}\mathbf{e}_z) - \zeta_1 (k_1^2 + k_2^2) - \Delta \zeta_1 \tag{2.2}$$

Here, Δ is the Laplace - Beltrami operator on the surface Γ and $B = \rho g L^2/\sigma_0$ is the Bond number.

In the case when $\lambda \ll \varepsilon^2$, the surface $s = \zeta_1(\varphi)$ satisfies Eq.(2.2), when $\lambda_0 = 0$, which is integrated separately from system (2.1). The case when $\lambda \gg \varepsilon^2$ is not considered.

3. Let us construct the selfsimilar solutions of system (2.1), (2.2) taking into account the axial symmetry subject to the condition that the gradient of the surface tension depends solely on the coordinate φ exponentially $d\sigma/d\varphi = \tau \varphi^n$. Let us now introduce the flow function by means of the relationships $h_{\varphi 0} = \partial \psi/\partial s$, $h_{\xi 1} = -r^{-1} \partial (r\psi)/\partial \varphi$ and use the notation $\eta =$

$s|\tau|^{1/3}\varphi^{(n-1/3)}$. We represent the functions ψ and q_0 in the form

$$\psi = |\tau|^{1/3}\varphi^{(n+2)/3}F(\eta),\quad q_0 = 3q\,|\tau|^{-4/3}\varphi^{(4n+2)/3}/(4n+2)$$

For the function $F(\eta)$ from system (2.1), we derive the equation

$$3F''' + (n+5)FF'' - (2n+1)F'^2 = 3q \tag{3.1}$$

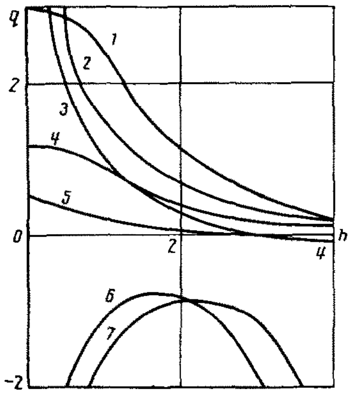

Fig.1

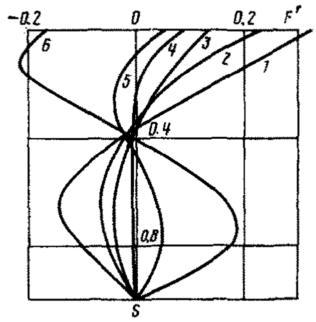

Fig.2

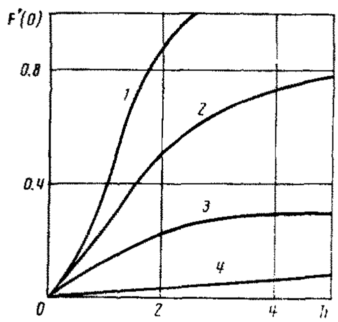

Fig.3

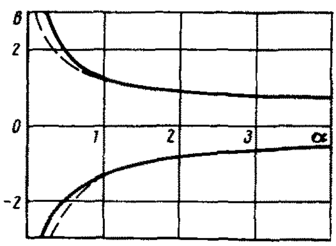

Fig.4

Let us now consider a layer of constant thickness $h = \text{const}$. A selfsimilar solution exists when $n = 1$. Placing the origin of the coordinate system on the free boundary, we write the boundary conditions for Eq.(3.1) as

$$F(0) = F(h) = F'(h) = 0,\quad F''(0) = -1 \tag{3.2}$$

We note that one of the conditions (3.2) serves to determine the unknown constant q which is proportional to the pressure in the layer. When $F''(0) = -1$, the tangential stresses on the free surface are directed towards an increase in the coordinate φ while, when $F''(0) = 1$, they are directed in the opposite direction.

When $n = 1$, problem (3.1), (3.2) was integrated numerically for various h. Curve 2 in Fig.1 shows the dependence $q(h)$. For small h, by expanding the function $F(\eta)$ in a series in powers of η, we find the asymptotic forms $q = 1.5\,h^{-1}[1 + o(1)]$ when $h \to 0$. When $h = 0.5$, the three significant figures are the same in the case of the asymptotic and numerical values of q. Curve 3 in Fig.2 shows the velocity profile when $h = 1$. The greatest velocity value is reached on the free boundary where the direction of the velocity is the same as that of the tangential stress. A counterflow zone with a maximum velocity which is approximately three times smaller than the velocity on the free boundary v_{Γ} arises close to the solid wall. As the thickness of the layer increases, the value of v_{Γ} and the presurface flow domain, where $h_{\varphi 0} > 0$ become smaller. Curve 4 in Fig.2 shows the velocity profile when $h = 10$. The dependence of $F'(0)$ on h is represented by curve 2 in Fig.3.

The asymptotic forms of the solution of problem (3.1), (3.2) are readily constructed when $h \to \infty$. We will present the principal terms of these asymptotic forms

$$q \sim h^{-2}G^2(\infty),\quad F(\eta) \sim h^{-2}\,G(\eta h)\quad (0 \leqslant \eta \leqslant \eta_0)$$

$$F(\eta) \sim h^{-2}G(\infty)(1-\eta)\quad (\eta_0 \leqslant \eta \leqslant 1),\ h \to \infty)$$

Here $\eta_0 = O(h^{-1})$ and the function $G(t)$ is determined from the boundary-value problem

$$G''' + 2GG'' - G'^2 = 0;\ G(0) = G'(\infty) = 0,\ G''(0) = -1$$

Numerical calculation shows that $G(\infty) = 0.7125$ and $G'(0) = 0.8972$.

The boundary-value problem (3.1), (3.2) was also solved numerically subject to the condition $F''(0) = 1$. A graph of the dependence $q(h)$ is shown by curve 7 in Fig.1. We note that asymptotic formulae are only constructed when $h \to 0:\ q \sim -1.5\,h^{-1},\ F \sim -0.5\,h\eta + 0{,}5h\eta^2 - 0.25\eta^3h^{-1}$.

In a layer with two free boundaries, the function $F(\eta)$ satisfies Eq.(3.1) when $n = 1$ with the boundary conditions $F(0) = F(h) = F''(h) = 0,\ F''(0) = \gamma\ (\gamma = \pm 1)$. The dependence $q(h)$ was calculated numerically and is shown graphically in Fig.1 where curves 3 and 6 correspond to values of γ equal to -1 and 1. The asymptotic formulae $q \approx -\gamma/h$ and $F'(0) \approx \gamma h/3$ can be found when $h \to 0$. We note that, unlike in the case of a layer with a solid boundary, the function $F'(\eta)$ is monotonic with a maximum on the free boundary.

Let us now construct the selfsimilar solutions in a thin layer lying on a horizontal solid surface when the free boundary has a point of contact with the solid wall at $\varphi = 0$. After linearization with respect to ζ_1, ζ_1' and ζ_1'' boundary condition (2.2) is reduced to the form

$$\lambda_0 q_0 = B\zeta_1 - \zeta_1'' - \zeta_1'/r \tag{3.3}$$

We will now consider the case when the term $B\zeta_1$ can be neglected. This is realized if the capillary forces exceed the gravitational forces, such as under conditions of weightlessness $(B = 0)$, for example. A selfsimilar solution only exists when $n = -7/5$. By writing the surface tension coefficient in the form $\sigma = \sigma_0 - 6.25\,\tau\varphi^{-2/5}$, we find the solution

$$\psi = -\varphi^{1/5}b^2\tau^{1/5}F(\eta),\quad \eta = 1 - s\varphi^{-4/5}b^{-1}\tau^{1/5},\quad \zeta_1 = b\tau^{-1/5}\varphi^{4/5}$$

The function $F(\eta)$ is determined from the boundary-value problem

$$5F''' + b^3(6FF'' + 3F'^2) = 5\alpha b^2 \tag{3.4}$$
$$F(0) = F(1) = F'(1) = 0,\ F''(0) = -1$$

The parameter b is proportional to the pressure gradient and has to be determined. The constant α is known: $\alpha = 96\lambda_0^{-1}\tau^{-6/5}/125$.

The boundary-value problem (3.4) was integrated numerically using the Runge-Kutta method. The dependence $b(\alpha)$ is represented by the solid lines in Fig.4. Two solutions were calculated numerically at fixed α. For one of these $b > 0$ and, for the second, $b < 0$. The asymptotic formulae for these solutions were constructed for large α by expanding the function $F(\eta)$ in a power series in η. We report the formula $b \approx \pm\sqrt{1.5/\alpha}\ (\alpha \to \infty)$.

We note that, when $\alpha = 1$, the three significant figures are the same in the case of the numerical and asymptotic values. When $\alpha \to 0$, the asymptotic forms of the solution of problem (3.4) were constructed for positive b:

$$b \approx c\alpha^{-1/5},\quad F \approx \alpha^{2/5}c^{-2}G\left(\frac{c\eta}{\alpha^{1/5}}\right)\qquad (0 \leqslant \eta \leqslant \eta_0)$$
$$F \approx -\alpha^{2/5}c^{-1/2}\sqrt{5/3}\,(\eta - 1)\quad (\eta_0 \leqslant \eta \leqslant 1,\quad \alpha \to 0)$$
$$\eta_0 = O(\alpha^{1/5}),\quad c = (\sqrt{0.6}\ G(\infty))^{2/5}$$

The function $G(t)$, describing the flow in the presurface layer, is determined from the boundary-value problem

$$5G''' + 6GG'' + 3G'^2 = 0;\quad G(0) = G'(\infty) = 0,\quad G''(0) = -1$$

It was found numerically that $G'(0) = 1.824$ and $G(\infty) = 1.543$.

As α becomes smaller, the velocity on the free boundary and the domain of presurface flow, where $h_{\varphi 0} > 0$, are reduced while there is a simultaneous increase in the thickness of the counterflow zone. The velocity profile when $\alpha = 10^{-3}$ passes close to curve 4 in Fig.2.

In the neighbourhood of the point of contact of the boundaries of the domain $(\varphi \to 0)$ the solution which has been obtained is invalid since the asymptotic expansion has a more complex form here than the form of (1.3) and the equations of the thin layer are the same as the complete Navier-Stokes system. An asymptotic investigation of the Navier-Stokes system close to the line of contact has been reported in /8, 9/.

With the boundary condition $F''(0) = 1$, a solution of problem (3.4) is obtained by replacing the variables F, α and b by $-F$, $-\alpha$ and $-b$ which has been considered above.

Let us now construct the selfsimilar solutions subject to the condition that the capillary forces in (3.3) can be neglected. We then write the condition on the free boundary in the form $\lambda_0 q_0 = B\zeta_1$. A selfsimilar solution now exists when $n = 4/5$ and this solution is written in the form $\psi = -\varphi^{-3/5} b^2 \tau^{1/5} F(\eta)$, where $\eta = 1 - s\varphi^{-2/5}\tau^{1/5}b^{-1}$ and $\zeta_1 = b\tau^{-1/5}\varphi^{2/5}$. The function F satisfies the boundary-value problem

$$5F''' + b^3(8FF'' - F^2) = 5\alpha b^2$$
$$F(0) = F(1) = F'(1) = 0, \quad F''(0) = -1$$

Here, $\alpha = B\lambda_0^{-1}\tau^{-6/5}$ is a specified parameter and the constant b is to be determined.

The solution $F(\eta)$ was calculated numerically for various α and the dependence $b(\alpha)$ is shown in Fig.4 by the broken lines. We present the asymptotic form $b \approx \pm\sqrt{1.5\alpha^{-1}}$ when $\alpha \to \infty$. The asymptotic formula $b \approx 0.5574\alpha^{-1/3}$ $(\alpha \to 0)$ was found for values of $b > 0$ at small α.

4. Let us now consider the case when a thermal flux $\varkappa \partial T/\partial n = Q$ is specified on the free boundary. Now, in order to satisfy the boundary condition for the tangential stress, it is necessary to calculate the temperature distribution within the layer.

We will derive the selfsimilar solution in the axially-symmetric case of a layer of constant height h when $Q = Q_0\varphi^2$, $T = \varphi^2\theta(s)$ and $\psi = \varphi F(s)$. The functions $F(s)$ and $\theta(s)$ are determined from the boundary-value problem

$$F''' + 2FF'' - F'^2 = q(h), \quad \theta'' + 2\mathrm{Pr}(\theta' F - \theta F') = 0 \qquad (4.1)$$
$$F(0) = 0, \quad F''(0) = -2\theta(0), \quad \theta'(0) = -1/\mathrm{Pr}$$
$$F(h) = F'(h) = \theta(h) = 0$$

Here, account has been taken of the fact that the quantity $Q_*/\varkappa$, where Q_* is the thermal flux scale, $\varkappa$ is the thermal conductivity and Pr is the Prandtl number, was adopted as the characteristic temperature scale.

Problem (4.1) was investigated numerically for different values of the parameters h and Pr. Curves *1*,*4* and *5* in Fig.1 represent the dependence of the pressure gradient on the thickness of the layer $q(h)$ at Prandtl numbers equal to 1, 7 and 50 respectively. We note that, unlike the problem when the temperature of the free boundary is specified $(\theta''(0) = -1)$, the function $q(h)$ has a finite value when $h = 0$.

We present the asymptotic formula for small h

$$q \sim 3/\sqrt{\mathrm{Pr}}, \quad F \sim {}^1/_2 (h^2 s - 2hs^2 + s^3)/\sqrt{\mathrm{Pr}}$$
$$\theta \sim (h - s)/\sqrt{\mathrm{Pr}} + {}^1/_2 h^3 s^2 \qquad (h \to 0)$$

Curves *1*, *3* and *4* in Fig.3 show the dependence of the velocity on the free boundary $F(0)$ on the thickness of the layer at Prandtl numbers of 1, 7 and 50 respectively. As h becomes larger, the velocity of the fluid increases and tends to a finite limit as $h \to \infty$. A pre-surface flow occurs in the domain close to the free boundary which is in the same direction as the tangential stresses together with a weaker counterflow zone close to the wall. The temperature on the free boundary decreases as the Prandtl number increases and tends to zero as $\mathrm{Pr} \to \infty$. When $h = 1$, the temperature profiles are close to being linear for $1 \leqslant \mathrm{Pr} \leqslant 50$.

5. Let us now consider the thermocapillary axially-symmetric flow in a layer of infinite depth with a specified gradient of the surface tension coefficient $d\sigma/d\varphi = \tau\varphi^n$. Here φ is the length of an arc on the free boundary in the meridional cross-section measured from the axis of symmetry. Specifying the flow function

$$\psi = \varphi^{(n+2)/3} \mid \tau \mid^{1/3} F(\eta), \quad \eta = s \mid \tau \mid^{1/3} \varphi^{(n-1)/3}$$

for $F(\eta)$, we derive Eq.(3.1) with the boundary conditions $F(0) = F''(\infty) = 0$ and $F''(0) = \pm 1$. The parameter q in (3.1) determines the amplitude of the pressure gradient in the external flow.

We now introduce the amplitude of the velocity on the free boundary $U_\Gamma = F'(0)$ and in the external flow $U_\infty = F'(\infty)$. The function $F(\eta)$ was calculated numerically for various n. The dependence of the amplitude of the velocity U_Γ on U_∞ when $n = 0$ is shown in Fig.5. Curve *1* corresponds to the case when $F''(0) = -1$ and curve *2* to the case when $F''(0) = 1$. Note that a single solution $F(\eta)$ is obtained for each value when $F''(0) = -1$ whereas two solutions exist for certain U_∞ when $F''(0) = 1$.

We present the asymptotic formula for large positive U_∞

$$U_\Gamma = U_\infty + \frac{a}{\sqrt{U_\infty}} + \ldots \qquad (U_\infty \to +\infty)$$

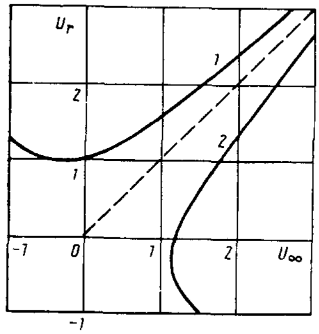

Fig.5

Here $a = G'(0)$, where $G(\eta)$ is determined from the boundary-value problem

$$3G''' + \eta(n+5)G'' - (4n+2)G' = 0$$
$$G'(\infty) = G(0) = 0, \quad G''(0) = \pm 1$$

When $n = 0$, we find $a = \mp 0.7743$ for $F''(0) = \pm 1$. Curve 2 in Fig.5 intersects the abscissa at $U_\infty = 1.084$. The boundary-value problem for $F(\eta)$ has two solutions close to this value. Note that the velocity profiles are monotonic in all cases. When the fluid is at rest $(U_\infty = 0)$ outside the boundary layer, no numerical solutions are found for $F''(0) = 1$, while, when $F''(0) = -1$, a single solution was calculated for which $F'(0) = 1.056$.

6. We will now consider the effect of buoyancy forces in a thin layer of an inhomogeneous fluid when there is a thermocapillary flow. The equations of motion are written in the Boussinesq approximation by replacing a term in the first equation of system (1.1) by the gravitational force $g\beta T e_z$, where β is the coefficient of thermal expansion. By considering a planar problem we find that a selfsimilar solution only exists when $d\sigma/d\varphi = \tau\varphi$. By representing the solution in the form

$$\psi = \varphi\,|\tau|^{1/3} F(\eta), \quad \theta = |\tau|\,\varphi^2\theta(\eta), \quad \eta = s\,|\tau|^{1/3}$$

we derive the boundary-value problem for $F(\eta)$ and $\theta(\eta)$

$$F^{(4)} + FF''' - F'F'' = \lambda\theta, \quad \theta'' + \Pr(F\theta' - 2F'\theta) = 0 \qquad (6.1)$$
$$F = 0,\ F'' = -1, \quad \theta = 1 \quad (\eta = 0), \quad F = F' = \theta = 0 \quad (\eta = h)$$

The parameter λ takes account of the effect of buoyancy forces in the thin layer. The sign of λ depends on the sign of the difference between the temperatures at the boundaries of the layer and $\lambda > 0$ if the free boundary is heated more strongly than the solid lower boundary.

Problem (6.1) was calculated numerically when $h = 1$ and $\Pr = 7$. The velocity profiles for various λ equal to 10,0, -15 and -30 are shown in Fig.2 (curves *1*, *2*, *5*, and *6* respectively). When $\lambda > 0$, there is an increase in the velocity on the free boundary v_Γ as λ increases and, as in a homogeneous fluid, the direction of the velocity is the same as the direction of the tangential stresses. For negative values of λ when $\lambda_1 \leqslant \lambda \leqslant 0$ $(\lambda_1 \approx -10)$, there is a retardation of the flow close to the free boundary. When $\lambda_2 \leqslant \lambda \leqslant \lambda_1$ $(\lambda_2 \approx -20)$, the flow domain subdivides into three zones, in two of which, which are adjacent to the boundaries of the domain, the velocity is positive while a counterflow occurs in the middle zone as shown by curve *5* in Fig.2, for example. When $\lambda < \lambda_2$, a reverse flow occurs close to the free boundary which is directed against the surface tangential forces and a counterflow (curve *6* in Fig.2) is formed in the remaining part of the flow.

Hence, buoyancy forces can lead to the reinforcement or attentuation of flows or to the formation of backward flows close to the free boundary.

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HYDRODYNAMIC SINGULARITIES IN FLOWS WITH A FREE BOUNDARY*

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A method of determining the shape of the free surface of a planar stationary flow of a ponderable ideal fluid which flows around point hydrodynamic singularities is proposed. A Cauchy problem is formulated for finding the profile of such a flow. The self-induced motion of a point vortex under the free surface of an ideal ponderous fluid is considered.

1. *The equation of the profile of a capillary-gravitational wave of small amplitude on the surface of a stationary flow.* In the case of a stationary flow having point singularities, a method was proposed in /1/ for finding the shape of the free surface when it deviates slightly from the unperturbed position. The solution obtained by this method has the form of an improper integral with a variable limit. For example, in the case of the flow round a vortex of intensity Γ located at a depth h by a flow having a velocity $-V$ at positive infinity, the following expression /2/ can be obtained for the shape of the free surface:

$$S(x) = -\frac{\Gamma}{\pi V}\int_{\infty}^{x} \frac{t\cos\nu(t-x) - h\sin\nu(t-x)}{t^2+h^2}\,dt \quad \left(\nu = \frac{g}{V^2}\right)$$

A still more complex integral representation for the function $S(x)$ can be obtained by this method when account is taken of capillary effects /3/.

Below, we obtain an ordinary differential equation which is satisfied by the function $S(x)$ and we formulate a Cauchy problem for determining it.

Let us consider a planar stationary flow with a velocity $-V$ at $x = \infty$. Let its unperturbed free surface coincide with the x-axis. Let us pick out the principal component of the flow by putting its complex potential equal to $W = \omega - Vz$, where $\omega = \varphi + i\psi$, $z = x + iy$. Linearized boundary conditions /3/

$$S(x) = (V/g)\,\varphi_x(x, 0) + [\alpha/(\rho g)]\,S''(x), \quad \psi(x, 0) = VS(x) \tag{1.1}$$

(α is the surface tension and ρ is the density of the fluid) can be written in the form of a single condition for the complex velocity $U(z) = w'(z)$ on the x-axis:

$$\operatorname{Im}(\beta U'' + iU' - \nu U) = 0 \quad (\beta = \alpha/(\rho V^2)) \tag{1.2}$$

Here and subsequently, a derivative of a function with respect to its argument is indicated by a prime.

If the flow passes around a unique singularity at the point $z_0 = -ih$, the complex velocity has the form $U(z) = C/(z+ih)^n + g(z)$, $n = 1, 2, \ldots$, where the function $g(z)$ is analytic over the whole of the domain of the flow. Following the method used in /1/, let us consider the function $f(z) = \beta U'' + iU' - \nu U$ which, as a consequence of condition (1.2), can be analytically extended according to the Schwartz principle into the upper half plane. Then,

$$f(z) = \beta F_+'' + iF_-' - \nu F_+, \quad F_\pm = C/(z+ih)^n \pm \bar{C}/(z-ih)^n$$

in the whole of the complex plane.

It can be directly verified that

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